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Phase dynamics of Bose-Einstein condensates: Losses versus revivals

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Abstract. In the absence of losses the phase of a Bose-Einstein condensate undergoes collapses and revivals in time due to elastic atomic interactions. As experiments necessarily involve inelastic collisions, we develop a model to describe the phase dynamics of the condensates in presence of collisional losses. We find that *a few inelastic processes* are sufficient to damp the revivals of the phase. For this reason the observability of phase revivals for present experimental conditions is limited to condensates with a few hundreds of atoms.

PACS. 03.75.Fi Phase coherent atomic ensemble (Bose condensation) – 05.30.Jp Boson systems

1 Introduction

Since the recent experimental observations of Bose-Einstein condensation in dilute atomic gases [1–5], much interest has been raised about the characteristic features of the condensate [6–8], and about its coherence properties in particular. Considerable attention has been devoted to the matter of the relative phase between two Bose-Einstein Condensates (BECs): how the phase manifests itself in an interference experiment (such as the one performed recently at MIT [9]), how the phase can be established by measurement, and how it evolves in presence of the elastic atomic interactions (see *e.g.* [10] and references therein). In this paper, in view of a possible experimental investigation of these problems, we complete the theoretical work already done on this subject by studying the dynamics of the relative phase in presence of loss processes occurring in the two condensates. Such loss processes, unavoidable in a real experiment, are due for example to collisions of condensed atoms with the background gas, or to three-body collisions between condensed atoms followed by recombination of two atoms to form a molecule [11,12].

We consider two mutually non interacting and spatially non overlapping BECs in two trapping potentials. We suppose that the experimentalist has at hand a device, such as the one depicted in Figure 1, allowing both the measurement of the relative phase between the condensates and the preparation of a state with a well-defined relative phase [13]. Starting from an initial state with a well-defined relative phase, we imagine that the two condensates evolve independently, under the influence of the atomic interactions, during a given time interval t at the end of which a measurement of the relative phase is performed. By repeating this procedure many times, one accesses the probability distribution of the relative phase [13].

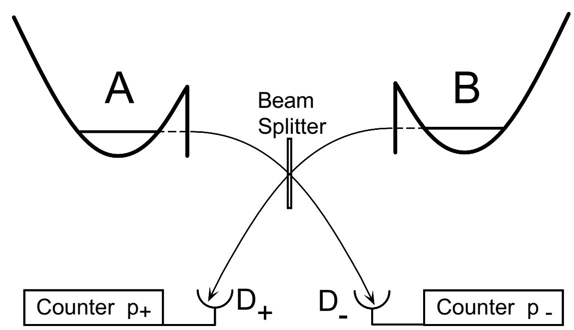

Fig. 1. Two BECs **A** and **B** in two non overlapping trapping potentials. Some atoms can be let out of the condensates towards a 50–50 atomic beam splitter. The detection of the atoms in the output channels of the beam-splitter realizes a measurement of the relative phase between the condensates.

In the lossless case, the relative phase shows *collapses* and *revivals* periodically in time due to the presence of elastic atomic interactions. In presence of losses, we find that *a few inelastic processes* are sufficient to dramatically damp the revivals of the phase. In practice, for typical experimental configurations, the observability of the revivals is limited to condensates with a small number of atoms, of the order of a few hundreds, for which the revival time is of the order of 0.1 to 1 second.

In Section 2 we present the theoretical model describing the evolution of the system in presence of losses. An interesting feature of the model is that it can be solved almost exactly analytically within the Monte-Carlo wave

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function approach recently put forward by several authors [14–17]. We take advantage of this circumstance in the following sections, to deduce analytical expressions for the interesting phase-dependent measurable quantities, and to a give a simple picture of the phase dynamics in presence of losses.

In Section 3 we find an approximate analytical expression for the evolution of a single stochastic wave function, and we give a simple physical interpretation of the result pointing out separately the role of the elastic atomic interactions and of the losses in the dynamics of the relative phase of the condensates. In Sections 4 and 5 we concentrate on the case in which the two condensates are placed in two identical traps and have initially the same average number of atoms, and we use the analytical results of Section 3 to calculate the time dependence of some relative phase dependent quantities. In particular in Section 4 we consider an interference experiment where one counts the atoms detected in the two output channels of the beam-splitter of Figure 1, and we analyze the two different physical situations in which the condensates' relative phase is initially sharply defined or is described by a "broad" relative phase distribution with a width $\gg 1/\sqrt{N}$. In Section 5 we imagine instead an experiment in which the time evolution of the whole relative phase probability distribution is measured. Sections 6 and 7 are dedicated to the analysis of additional features that would appear in an experiment; the effect of asymmetries in the parameters of the two condensates and in the initial average number of atoms is considered in Section 6, and the effect of fluctuations in the initial total number of atoms is considered in Section 7. Some concluding remarks are presented in Section 8.

2 Model

2.1 Master equation

Let us consider two mutually non-interacting and spatially non-overlapping BECs **A** and **B** in two harmonic potentials. Our starting point to describe the evolution of this system in presence of m-body losses is a master equation for the density matrix ρ describing the atoms in the traps:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho] + \int d^3\mathbf{r}\,\kappa\big[[\hat\psi(\mathbf{r})]^m\,\rho\,[\hat\psi^\dagger(\mathbf{r})]^m - \frac{1}{2}\{[\hat\psi^\dagger(\mathbf{r})]^m[\hat\psi(\mathbf{r})]^m,\rho\}\big], \quad (1)$$

where $\{X,Y\}$ denotes the anticommutator, and $[\hat\psi(\mathbf{r})]^m$ is the field operator raised to the power m which suppresses m particles in $\mathbf{r}$. In second quantized form the Hamiltonian H reads:

$$H = \int d^3\mathbf{r}\left[\hat\psi^\dagger(\mathbf{r})H_0\hat\psi(\mathbf{r}) + \frac{g}{2}\hat\psi^\dagger(\mathbf{r})\hat\psi^\dagger(\mathbf{r})\hat\psi(\mathbf{r})\hat\psi(\mathbf{r})\right], \quad (2)$$

where H_0 is the one-particle Hamiltonian including the trapping potential and the kinetic energy, and $g = 4\pi\hbar^2 a/M$ where M is the mass of the atoms and a is the s-wave scattering length.

The loss terms in equation (1) are parameterized by the number m of particles lost per collisional event and by the collisional constant κ. Physically the case $m = 1$ corresponds to collisions of atoms in the condensate with atoms of background gas in the cell; the case $m = 2$ corresponds to spin-flip collisions between condensed atoms in magnetic traps, as only specific spin components are trapped; the case $m = 3$ corresponds to three-body collisions between condensed atoms, leading to the formation of an excited molecule and a hot atom supposed to leave the condensate. The collisional constant κ for the processes $m = 1$ and $m = 3$ has been measured for ^{87}Rb atoms at JILA [11] and for ^{23}Na atoms at MIT [12]. The collisional constant for the $m = 2$ process has not been accurately measured for these atoms yet, as the two-body losses seem to give a smaller contribution to the total decay rate.

We assume that at any time the state of the condensate **A** (resp. **B**) can be described in terms of a single occupied mode, neglecting the excitations out of this mode due to a non-zero temperature or to the loss processes. We assume furthermore that these modes are the single particle ground state wave functions ϕ_a, ϕ_b given self-consistently as functions of the number of particles by the Gross-Pitaevskii equation:

$$\left[H_0 + gN_\epsilon|\phi_\epsilon(\mathbf{r};N_\epsilon)|^2\right]\phi_\epsilon(\mathbf{r};N_\epsilon) = \mu_\epsilon(N_\epsilon)\phi_\epsilon(\mathbf{r};N_\epsilon), \quad (3)$$

where the $\mu_\epsilon(N_\epsilon)$'s are the chemical potentials for the condensates with N_ϵ particles, and where the wave functions ϕ_ϵ are normalized to unity. In more mathematical words we approximate the atomic field operator by:

$$\hat\psi(\mathbf{r}) = \sum_{\epsilon=a,b} c_\epsilon\phi_\epsilon(\mathbf{r};\hat N_\epsilon) \quad (4)$$

where the operators $c_a^\dagger$ ($c_b^\dagger$) and c_a (c_b) create and annihilate a particle in the condensate **A** (**B**) respectively, and where $\hat N_\epsilon = c_\epsilon^\dagger c_\epsilon$ are the operators giving the number of particles in each condensate. Note that we keep in equation (4) the dependence of the mode on the number of particles in the condensate.

By substituting equation (4) into equation (2) we get

$$H = E_a(\hat N_a) + E_b(\hat N_b) \quad (5)$$

with

$$E_\epsilon(N_\epsilon) = N_\epsilon\left[\int d^3\mathbf{r}\phi_\epsilon^*(\mathbf{r};N_\epsilon)H_0\phi_\epsilon(\mathbf{r};N_\epsilon) + \frac{gN_\epsilon}{2}|\phi_\epsilon(\mathbf{r};N_\epsilon)|^4\right] \quad (6)$$

(we have used $N_\epsilon - 1 \simeq N_\epsilon$).

By assuming that in the considered time interval the atom number distributions in the two condensates remain peaked around the initial average values:

$$\bar N_\epsilon = Tr[\rho(0)c_\epsilon^\dagger c_\epsilon], \quad (7)$$

we expand the condensates' Hamiltonian around $\bar{N}_a$, $\bar{N}_b$ keeping up to the quadratic terms:

$$H(\hat{N}_a, \hat{N}_b) \simeq H^q(\hat{N}_a, \hat{N}_b) \equiv \sum_{\epsilon=a,b} E(\bar{N}_\epsilon) + (\hat{N}_\epsilon - \bar{N}_\epsilon) \times \mu_\epsilon(\bar{N}_\epsilon) + \frac{1}{2}(\hat{N}_\epsilon - \bar{N}_\epsilon)^2 \mu'_\epsilon(\bar{N}_\epsilon). \quad (8)$$

In our model we will use this quadratic version of the Hamiltonian, where the chemical potentials μ_a and μ_b and their derivatives can be calculated by solving numerically the Gross-Pitaevskii equation (3).

We now substitute our ansatz equation (4) in the loss part of the master equation; since the condensates do not overlap this amounts to the substitution

$$[\hat{\psi}(\mathbf{r})]^m \to \sum_{\epsilon=a,b} [c_\epsilon \phi_\epsilon(\mathbf{r}; \hat{N}_\epsilon)]^m \quad (9)$$

in equation (1). In contrast to the Hamiltonian part which required a careful quadratization in $\hat{N}_\epsilon - \bar{N}_\epsilon$ to get the correct phase dynamics, the dissipative part will be treated to lowest order by replacing $\hat{N}_\epsilon$ by $\bar{N}_\epsilon$ in equation (9). This allows us finally to obtain a master equation of the form:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H^q(\hat{N}_a, \hat{N}_b), \rho] + \sum_{\epsilon=a,b} \gamma_\epsilon [c_\epsilon]^m \rho [c_\epsilon^\dagger]^m - \frac{\gamma_\epsilon}{2}\{[c_\epsilon^\dagger]^m [c_\epsilon]^m, \rho\}, \quad (10)$$

where (for $\epsilon = a, b$) we have introduced the rates for the m-body collisions:

$$\gamma_\epsilon = \kappa \int d^3\mathbf{r} |\phi_\epsilon(\mathbf{r}; \bar{N}_\epsilon)|^{2m}. \quad (11)$$

2.2 Stochastic formulation

To study the evolution of the system we adopt the Monte-Carlo wave function point of view [14] which provides us with a stochastic formulation of the master equation (10). To this aim we introduce the jump operators:

$$S_\epsilon = \sqrt{\gamma_\epsilon}[c_\epsilon]^m \qquad \epsilon = a, b \quad (12)$$

and an effective Hamiltonian:

$$H_{eff} = H^q - \frac{i\hbar}{2} \sum_{\epsilon=a,b} S_\epsilon^\dagger S_\epsilon \, . \quad (13)$$

The Monte-Carlo wave function $|\psi(t)\rangle$ undergoes a non hermitian Hamiltonian evolution ruled by H_{eff} (plus a continuous renormalization) interrupted by random quantum jumps occurring at a rate $\langle\psi(t)|\sum_{\epsilon=a,b}(S_\epsilon^\dagger S_\epsilon)|\psi(t)\rangle$, where $|\psi(t)\rangle$ is normalized to unity. The effect of a quantum jump is to replace $|\psi\rangle$ by $S_\epsilon|\psi\rangle$ up to a normalization factor. Physically this corresponds to the loss of m particles in the condensate ϵ *via* the m-body collisional processes described above. The two kinds of jumps $\epsilon = a, b$ occur with relative probabilities:

$$\frac{P_a}{P_b} = \frac{\langle\psi(t)|S_a^\dagger S_a|\psi(t)\rangle}{\langle\psi(t)|S_b^\dagger S_b|\psi(t)\rangle} \, . \quad (14)$$

Starting with a state with a fixed total number of particles N, we can expand at each time the state vector on the Fock basis

$$|\psi(t)\rangle = \sum_{N_a=0,\tilde{N}} d_{N_a} |N_a, \tilde{N} - N_a\rangle, \quad (15)$$

where $\tilde{N}$ is the total number of atoms at time t in the two condensates, and we can carry out the evolution numerically. The mean value of an observable $\hat{O}$ is obtained by averaging the expectation value $\langle\psi(t)|\hat{O}|\psi(t)\rangle$ over all possible stochastic realizations for the evolution of $|\psi(t)\rangle$.

Usually the Monte-Carlo wave function technique is carried out purely numerically. It turns out that for the present problem it is possible to treat analytically the evolution of a Monte-Carlo wave function and, after a minor approximation, average analytically over all the possible stochastic realizations. This leads to a simple interpretation of the dynamics and allows the derivation of analytical formulas for observables' mean values. As it will appear in the figures the analytical results are in good agreement with the numerical results.

3 Evolution of a single wave function

In this section we derive an approximate formula for the evolution of a single stochastic wave function, and we discuss its physical interpretation. We first consider the simple case in which the condensates are initially in a *phase state*, introduced in the beginning of the section, and subsequently the general case in which the initial state is characterized by a given relative phase distribution.

For the following it will be useful to introduce the operators

$$\hat{N} = \hat{N}_b + \hat{N}_a, \quad \hat{n} = \hat{N}_b - \hat{N}_a \quad (16)$$

corresponding to the sum and difference of the number of atoms in **A** and in **B**.

3.1 Phase states

A very useful class of states of two condensates is represented by the *phase states* [18]:

$$|\phi\rangle_N = \frac{1}{\sqrt{2^N N!}} \, (c_a^\dagger e^{i\phi} + c_b^\dagger e^{-i\phi})^N \, |0\rangle \quad (17)$$

having a fixed total number of particles N and leading to a well-defined relative phase 2ϕ between the condensates **A** and **B**. These states have the remarkable properties:

$$c_\epsilon|\phi\rangle_N = \sqrt{\frac{N}{2}} e^{i\phi\,(\delta_{\epsilon,a}-\delta_{\epsilon,b})}\;|\phi\rangle_{N-1} \quad \epsilon = a,b \tag{18}$$

$$e^{-i\alpha\hat{n}}|\phi\rangle_N = |\phi+\alpha\rangle_N \qquad \forall\alpha, \tag{19}$$

where the $\delta_{\epsilon,\epsilon'}$ for $\epsilon,\epsilon' = a,b$ are Kronecker deltas. The first property reflects the fact that in a phase state, all the particles are in the same state (see Eq. (17)), and the second one shows that n and ϕ are to some extent conjugate variables like the momentum and position of a particle. Note that the phase states are not orthogonal:

$$_N\langle\phi'|\phi\rangle_N = [\cos(\phi-\phi')]^N, \tag{20}$$

though the function $[\cos(\phi-\phi')]^N$ in equation (20) becomes very peaked around zero when $N\to\infty$ with a width scaling as $1/\sqrt{N}$. Any state with a total number N of particles can be expanded on the overcomplete set of phase states:

$$|\psi\rangle = \mathcal{A}\int_{-\pi/2}^{\pi/2}\frac{d\phi}{\pi}\;c(\phi)\;|\phi\rangle_N, \tag{21}$$

where $c(\phi)$ can be obtained from the expansion of the state vector on the Fock state basis:

$$c(\phi) = \mathcal{A}^{-1}\sum_{N_a=0,N} 2^{N/2}\left(\frac{N_a!(N-N_a)!}{N!}\right)^{1/2} \times e^{i(N-2N_a)\phi}\;\langle N_a, N-N_a|\psi\rangle. \tag{22}$$

The quantity $|c(\phi)|^2$ can be interpreted as the relative phase probability distribution [13]. This distribution, flat for a Fock state and very peaked for a phase state, is normalized in such a way that:

$$\int_{-\pi/2}^{\pi/2}\frac{d\phi}{\pi}\;|c(\phi)|^2 = 1. \tag{23}$$

The factor $\mathcal{A}$ in equation (21) ensures that $|\psi\rangle$ is normalized to unity. For $N\gg 1$ and for a $c(\phi)$ varying slowly at the scale $1/\sqrt{N}$, we can replace the scalar product $_N\langle\phi'|\phi\rangle_N$ by the delta distribution $\sqrt{2\pi/N}\delta(\phi-\phi')$ to obtain $\mathcal{A} = (\pi N/2)^{1/4}$.

3.2 Approximate expression for $|\psi(t)\rangle$

Consider the evolution of the state vector $|\psi(t)\rangle$, from a time $t_0 = 0$ to a time t, for *a particular stochastic realization.* We imagine that k quantum jumps, each corresponding to the loss of m particles, occur at times $t_1,...,t_k$ separated by time intervals $\tau_j = t_j - t_{j-1}$ with $j = 1,...,k$; the kth jump takes place in the condensate ϵ_k with $\epsilon_k = a,b$. We have:

$$|\psi(t)\rangle = \mathcal{N}e^{-\frac{i}{\hbar}H_{eff}(t-t_k)}S_{\epsilon_k}e^{-\frac{i}{\hbar}H_{eff}\tau_k}S_{\epsilon_{k-1}} \times e^{-\frac{i}{\hbar}H_{eff}\tau_{k-1}}...\,S_{\epsilon_1}e^{-\frac{i}{\hbar}H_{eff}\tau_1}|\psi(0)\rangle \tag{24}$$

where $\mathcal{N}$ is a normalization factor. By using the identity:

$$[c_\epsilon]^m f(\hat{N}_a,\hat{N}_b) = f(\hat{N}_a + m\delta_{\epsilon,a},\hat{N}_b + m\delta_{\epsilon,a})\,[c_\epsilon]^m \qquad \epsilon = a,b, \tag{25}$$

we shift all the jump operators in equation (24) to the right by letting them "pass through" the exponentials and we obtain:

$$\begin{aligned}|\psi(t)\rangle = \mathcal{N}&\exp[-iH_{eff}(\{\hat{N}_\epsilon\})(t-t_k)/\hbar]\\ &\times\exp[-iH_{eff}(\{\hat{N}_\epsilon + m\delta_{\epsilon,\epsilon_k}\})\tau_k/\hbar]\\ &\times\exp[-iH_{eff}(\{\hat{N}_\epsilon + m(\delta_{\epsilon,\epsilon_k}+\delta_{\epsilon,\epsilon_{k-1}}\})\tau_{k-1}/\hbar]...\\ &\times\prod_{j=1,k}S_{\epsilon_j}|\psi(0)\rangle.\end{aligned} \tag{26}$$

We introduce now the major approximation in our calculations by replacing $[c_\epsilon^\dagger]^m[c_\epsilon]^m$ by $\bar{N}_\epsilon^{\;m}$ in the expression for the effective Hamiltonian equation (13), supposing that the fraction of lost particles is small. The resulting effective Hamiltonian then takes the form:

$$H_{eff} = H^q - \frac{i\hbar}{2}\lambda\,, \tag{27}$$

quadratic in $\hat{N}_a$ and $\hat{N}_b$, where λ is a constant representing the mean total number of collisional events per unit of time:

$$\lambda = \lambda_a + \lambda_b \quad \text{with} \quad \lambda_a = \gamma_a\bar{N}_a^{\;m}, \quad \lambda_b = \gamma_b\bar{N}_b^{\;m}. \tag{28}$$

In this approximation the statistics of the quantum jumps is simply Poissonian with a parameter λ and $\delta_{b,\epsilon_j} = 1 - \delta_{a,\epsilon_j}$ takes the values 1 and 0 with probabilities λ_b/λ and λ_a/λ respectively, according to equation (14).

We then expand the effective Hamiltonians in each exponential in equation (26) around $\hat{N}_a$, $\hat{N}_b$ in powers of $m\delta_{\epsilon,\epsilon_k}$, $m(\delta_{\epsilon,\epsilon_k}+\delta_{\epsilon,\epsilon_{k-1}})$, *etc.* Due to the quadratic dependence of equation (27) on $\hat{N}_a$ and $\hat{N}_b$ we limit the expansion at the first order, the subsequent terms being constants or zero. By using equation (27) we then obtain the following result for the state vector at time t:

$$|\psi(t)\rangle = \mathcal{N}e^{-\lambda t/2}U_0(t)U_1(t)\prod_{j=1,k}S_{\epsilon_j}|\psi(0)\rangle. \tag{29}$$

In equation (29) we have introduced the unitary operators

$$U_0(t) = \exp[-iH^q(\{\hat{N}_\epsilon\})t/\hbar] \tag{30}$$

$$U_1(t) = \exp\left[-i\left(\frac{\partial H^q}{\partial N_a}(\{\hat{N}_\epsilon\})\Delta_a + \frac{\partial H^q}{\partial N_b}(\{\hat{N}_\epsilon\})\Delta_b\right)/\hbar\right] \tag{31}$$

where for $\epsilon = a,b$:

$$\Delta_\epsilon = m\sum_{j=1,k}\sum_{l=j,k}\delta_{\epsilon,\epsilon_l}\tau_j = m\sum_{l=1,k}\delta_{\epsilon,\epsilon_l}t_l \tag{32}$$

are random quantities that depend on the particular realization.

We sketch out briefly the physical interpretation of the result equation (29), considering the action of the successive factors in equation (29) on a *phase state* defined in equation (17).

- The factor $U_0(t)$ in equation (29) accounts for the evolution in absence of losses. Expressed in terms of the operators $\hat{N}$ and $\hat{n}$ of equation (16) it involves:

$$H^q(\{N_\epsilon\}) = f_0(\hat{N}) + \hat{n}v(\hat{N}) + \hat{n}^2(\mu_b' + \mu_a')/8. \quad (33)$$

We have used equation (8) and we have defined

$$v(\hat{N}) = \frac{1}{2\hbar}\{\mu_b - \mu_a + \frac{\mu_b' - \mu_a'}{2}(\hat{N} - \bar{N}) - \frac{\mu_b' + \mu_a'}{2}(\bar{N}_b - \bar{N}_a)\}, \quad (34)$$

where $\bar{N} = \bar{N}_a + \bar{N}_b$ and where μ_ϵ stands for $\mu_\epsilon(\bar{N}_\epsilon)$. From the properties of the phase state we find that the terms in $\hat{n}$ and $\hat{n}^2$ in equation (33), when exponentiated in U_0, (i) shift the relative phase at the N-dependent constant speed $v(\hat{N})$ and (ii) spread the relative phase (in a way analogous to the spreading of a wave packet of a massive particle under free evolution), respectively. The term $f_0(\hat{N})$ in equation (33) is a function of the total number of atoms N only and plays no role, since it amounts in $U_0(t)$ to adding a global phase factor to the wave function. The phase-spreading will eventually lead to a *collapse* of the relative phase [6]. On the other hand due to the discreteness of the spectrum of the operator $\hat{n}$ (the spectrum of $\hat{n}$ consists of even integers for an even N, and of odd integers for an odd N), there are special times at which the exponential operator equation (33) reduces to a mere translation of the relative phase, yielding the well-known result that *revivals* should follow the collapses of the relative phase. More precisely if one uses the expansion equation (15) for the phase state defined in equation (17), one realizes that a relative phase distribution initially peaked around ϕ_0 displays revivals at the times:

$$t_R = q\pi/\chi, \quad q \text{ integer} \quad (35)$$

where we have introduced:

$$\chi = \frac{\mu_a' + \mu_b'}{2\hbar}\,. \quad (36)$$

At these times, for N even:

$$e^{-i\chi\hat{n}^2 t_R/4}|\phi\rangle_N = |\phi + q\pi/2\rangle_N \quad (37)$$

and for N odd:

$$e^{-i\chi\hat{n}^2 t_R/4}|\phi\rangle_N = e^{-iq\pi/4}|\phi\rangle_N. \quad (38)$$

The initial relative phase distribution is then reconstructed around $(\phi_0 + v(N)t_R + q\pi/2)$ for N even and around $(\phi_0 + v(N)t_R)$ for N odd.

- The factor $U_1(t)$ in equation (29) accounts for the presence of losses. Expressed in terms of the operators $\hat{n}$ and $\hat{N}$, it involves:

$$\frac{\partial H^q}{\partial N_a}(\{N_\epsilon\})\,\Delta_a/\hbar + \frac{\partial H^q}{\partial N_b}(\{N_\epsilon\})\,\Delta_b/\hbar = f_1(\hat{N}) + \hat{n}D \quad (39)$$

where global phase factors are included in $f_1(\hat{N})$. The translation operator $\hat{n}$ appears in equation (39) multiplied by a random quantity D defined as:

$$D = m\sum_{l=1,k} t_l \left[\chi\delta_{b,\epsilon_l} - \frac{\mu_a'}{2\hbar}\right]. \quad (40)$$

Equations (19, 39) show that the relative phase in a single stochastic realization is shifted by the random amount D due to the loss processes. This effect will turn out to have a dramatic influence on the coherence properties of the condensates.

- Finally in equation (29) the action of the jump operators on a phase state is simply:

$$\prod_{j=1,k} S_{\epsilon_j}|\phi\rangle_N = \left[\frac{N}{2}\frac{N-1}{2}\dots\frac{N-mk+1}{2}\right]^{1/2} \times e^{-i\phi\alpha}|\phi\rangle_{N-mk} \quad (41)$$

where we have introduced the quantity

$$\alpha = m\sum_{j=1,k}\left[2\delta_{b,\epsilon_j} - 1\right]. \quad (42)$$

Apart from numerical factors that will be absorbed in the normalization and the phase factor involving α, equation (41) amounts to reducing by a random amount the total number of particles.

In the general case, an initial state with N particles can be expanded on the phase states set (see Eq. (21)). By using equations (33, 39, 41), and getting rid of the global phase factors we then obtain the wave function:

$$|\psi(t)\rangle = \mathcal{B}(t)\int_{-\pi/2}^{\pi/2}\frac{d\phi}{\pi}c(\phi,0)e^{-i\chi\hat{n}^2 t/4} \times e^{-i\phi\alpha}|\phi + D + v(N-mk)t\rangle_{N-mk}, \quad (43)$$

where $\mathcal{B}(t)$ is a normalization factor.

4 Mean beating intensity of the condensates

To monitor the evolution of the relative phase between the condensates, a possible choice is to determine the relative phase dependent quantity $\langle c_a^\dagger c_b\rangle$ after some time during which the two condensates, initially prepared in a state with a defined relative phase, evolve independently. As the relative phase between the condensates is affected by the elastic atomic interactions, the average $\langle c_a^\dagger c_b\rangle$ undergoes collapses and revivals in time.

In the situation described in Figure 1 the measure of $\langle c_a^\dagger c_b\rangle$ would correspond to the following measurement scheme: Prepare a state in which **A** and **B** have a well-defined relative phase [13]; let the condensates evolve during a time interval t; then let $p \ll N$ atoms escape from the condensates and beat them on the beam-splitter. The counts registered in the two output channels of the beam-splitter will be fluctuating variables whose averages over many realizations of the whole procedure are [13]:

$$I_{\pm} = \langle \frac{p}{\hat{N}} \frac{(c_a^\dagger \pm c_b^\dagger)(c_a \pm c_b)}{2} \rangle \simeq \frac{p}{\bar{N}} \frac{1}{2} \left(\langle c_a^\dagger c_a \rangle + \langle c_b^\dagger c_b \rangle \pm 2Re \langle c_a^\dagger c_b \rangle \right), \quad (44)$$

the difference between I_+ and I_- gives then the real part of $\langle c_a^\dagger c_b \rangle$.

We shall now use the approximated formulas (29, 43) to calculate the time dependence of $\langle c_a^\dagger c_b \rangle$. The main result of this section is that the revivals in this quantity are damped in time with a simple exponential law $e^{-\lambda t}$ where the constant λ, defined in equation (28), is the mean number of loss processes per unit of time.

In the present and in the following section we restrict for simplicity to the perfectly *symmetric* case where the two trapping potentials are identical and the two condensates have initially the same mean number of particles:

$$\bar{N}_a = \bar{N}_b, \quad (45)$$

$$\gamma_a = \gamma_b, \quad (46)$$

$$\mu_a = \mu_b. \quad (47)$$

Moreover we consider an initial state having a *fixed* total number of particles equal to N; and as a reminder of this choice (when it is the case) we will attach a superscript $\langle ... \rangle^{fix}$ to the averages. The *non symmetric* case for the condensates will be considered in Section 6; while the effect of fluctuations in the initial total number of atoms (requiring a further averaging over N) will be analyzed in Section 7.

We calculate $\langle c_a^\dagger c_b \rangle^{fix}$ in two different physical situations. The first one refers to a sharply defined initial relative phase ($\Delta\phi \simeq 1/\sqrt{N}$) for which we choose a *phase state* as the initial state; the second one, probably more realistic from the experimental point of view, makes use of an initial phase distribution much broader than $1/\sqrt{N}$. In each case we first calculate the expectation value of the operator $\hat{O} = c_a^\dagger c_b$ for a single stochastic realization using the results of Section 3, and then take the average over the stochastic realizations. In the whole paper we will denote with $\langle \psi(t)|\hat{O}|\psi(t)\rangle$ the single realization expectation value and with $\langle \hat{O} \rangle$ the quantum mechanical average.

4.1 Case of an initial phase state

Let us assume $|\psi(0)\rangle = |\phi\rangle_N$; by using equations (29) and (33, 39, 41), for a single realization, we find:

$$\langle \psi(t)|c_a^\dagger c_b|\psi(t)\rangle = {}_{N-mk}\langle \phi + D| e^{i\frac{\chi}{4}\hat{n}^2 t} c_a^\dagger c_b \times e^{-i\frac{\chi}{4}\hat{n}^2 t} |\phi + D\rangle_{N-mk} \quad (48)$$

where χ and D are defined in equation (36) and equation (40) respectively. Note that the contribution involving the drift velocity of equation (34) vanishes as we are considering here the *symmetric* case. The quadratic dependence on $\hat{n}$ in equation (48) can be eliminated by shifting $c_a^\dagger c_b$ through the exponential $e^{-i\frac{\chi}{4}\hat{n}^2 t}$ using equation (25):

$$e^{i\frac{\chi}{4}\hat{n}^2 t} c_a^\dagger c_b e^{-i\frac{\chi}{4}\hat{n}^2 t} = e^{-i\chi(\hat{n}+1)t} c_a^\dagger c_b \quad (49)$$

so that

$$\langle \psi(t)|c_a^\dagger c_b|\psi(t)\rangle = {}_{N-mk}\langle \phi + D| e^{-i\chi(\hat{n}+1)t} \times c_a^\dagger c_b |\phi + D\rangle_{N-mk}; \quad (50)$$

by using the properties (18, 19, 20) we then have:

$$\langle \psi(t)|c_a^\dagger c_b|\psi(t)\rangle = \frac{N-mk}{2} e^{-2i\phi} e^{-2iD} \times [\cos(\chi t)]^{N-mk-1}. \quad (51)$$

The next step is to take the average of the result equation (51) over the stochastic realizations which amounts to averaging over the random variables k, τ_j and δ_{b,ϵ_j} (the last two variables appearing in the random quantity D). We show the calculation of the average in detail in the Appendix A. The result for $\langle c_a^\dagger c_b \rangle^{fix}$ reads:

$$\langle c_a^\dagger c_b \rangle^{fix} = e^{-2i\phi} e^{-\lambda t} \sum_{k=0,N/m-1} \frac{N-mk}{2} \frac{1}{k!} \times [\lambda t \, u(t)]^k [\cos(\chi t)]^{N-mk-1}, \quad (52)$$

where the function $u(t)$ is given by:

$$u(t) = \frac{\sin(m\chi t)}{m\chi t}. \quad (53)$$

By identifying the factor $N - mk$ with N under the assumption of a small fraction of lost particles, and by extending the sum over k up to ∞, we are able to express the result in a compact way[1]:

$$\langle c_a^\dagger c_b \rangle^{fix} = e^{-2i\phi} e^{-\lambda t[1-u(t)/\cos^m(\chi t)]} \frac{N}{2} [\cos(\chi t)]^{N-1}. \quad (54)$$

The factor $[\cos(\chi t)]^{N-1}$ in equation (54), already obtained in [19] in the absence of losses, is responsible for the collapses of the average value $\langle c_a^\dagger c_b \rangle^{fix}$ and for revivals at times $t_R = q\pi/\chi$ with q integer. The collapses and revivals of $\langle c_a^\dagger c_b \rangle^{fix}$ are shown in Figure 2 both (a) in absence and (b) in presence of three-body losses. We see immediately that the losses have a dramatic effect reducing exponentially in time the average with the rate λ given by equation (28). In fact at a revival times $t = t_R$, $u(t)$ vanishes so that the average value of $\langle c_a^\dagger c_b \rangle^{fix}$ is simply attenuated with respect to the lossless case:

$$\langle c_a^\dagger c_b \rangle_{t=t_R}^{fix} = (-1)^{q(N-1)} \langle c_a^\dagger c_b \rangle_{t=0}^{fix} \, e^{-\lambda t_R}, \quad (55)$$

[1] It should be noted however that the compact formula (54) diverges for $\chi t = \pi/2 + q\pi$, where the explicit sum equation (52) should be used instead. At such points $\langle c_a^\dagger c_b \rangle^{fix} = 0$ anyway.

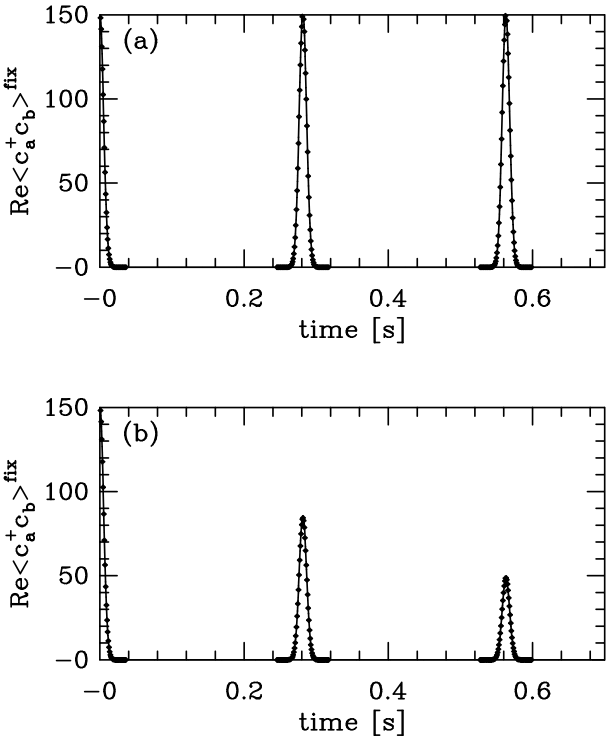

Fig. 2. Collapses and revivals of $\langle c_a^\dagger c_b\rangle^{fix}$ for an initial phase state (a) without losses and (b) in presence of 3-body losses. The calculation is performed for ^{87}Rb atoms in the $F = 1, m_F = -1$ state and for isotropic harmonic traps. The 3-body loss rate is inferred from the experimental data of JILA. The initial total number of atoms is $N = 301$, and the harmonic frequencies are $\Omega_a/2\pi = \Omega_b/2\pi = 500$ Hz. Diamonds: numerical result with 2.5×10^4 Monte-Carlo wave functions. Solid line: analytical result.

by an exponential factor which is exactly the probability that no particles are lost up to time t. The effect of losses on the revivals, already significative when $\lambda t_R \simeq 1$ (that is *one* loss process has occurred on average at the revival time), can be understood by the fact that in each single Monte-Carlo realization experiencing a quantum jump at a time $t \sim t_R$ the relative phase is shifted by an amount $D \gtrsim \pi$. This point will be further exemplified in Section 5.

4.2 Case of an initial relative phase distribution broader than that of a phase state

Since it may be difficult to prepare experimentally the condensates in a phase state we now consider the more realistic case in which the initial relative phase distribution $|c(\phi,0)|^2$ for the condensates is broad as compared to $1/\sqrt{N}$. To be specific we assume that the initial relative phase distribution is a Gaussian centered at $\phi = 0$:

$$c(\phi,0) = \mathcal{G}_0 \exp\left(-\phi^2/(4\Delta\phi^2)\right) \qquad \frac{1}{\sqrt{N}} \ll \Delta\phi \ll 1\,, \quad (56)$$

where ϕ ranges between $-\pi/2$ and $\pi/2$. This choice corresponds to a Gaussian distribution for the number of particles in the condensates:

$$\langle N_a, N - N_a|\psi(0)\rangle = \mathcal{G} e^{-(N-2N_a)^2/4\Delta n^2} \qquad (57)$$

with $\Delta n\, \Delta\phi = 1/2$.

For a single realization, we use equation (43) and we proceed along the lines of the previous calculation to get:

$$\langle\psi(t)|c_a^\dagger c_b|\psi(t)\rangle = \left[\frac{\pi\tilde{N}}{2}\right]^{1/2} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi}\frac{d\phi'}{\pi} \times c(\phi,0)c^*(\phi',0)\frac{\tilde{N}}{2}e^{-i\alpha(\phi-\phi')} \times e^{-i(\phi+\phi'+2D)}\,{}_{\tilde{N}-1}\langle\phi' - \chi t|\phi\rangle_{\tilde{N}-1} \qquad (58)$$

where $\tilde{N} = N - mk$ with k equal to the number of quantum jumps experienced by the Monte-Carlo wave function up to time t. Now by using the fact that the scalar product between the phase states for $N \gg 1$ is a very peaked function of $\phi - \phi'$ with respect to the other functions in the integral, we perform the substitution:

$${}_{\tilde{N}-1}\langle\phi' - \chi t|\phi\rangle_{\tilde{N}-1} \to \cos^{\tilde{N}-1}(q_0\pi)\sqrt{\frac{2\pi}{\tilde{N}}} \times \delta(\phi' + q_0\pi - \chi t - \phi) \qquad (59)$$

where the integer q_0 is chosen such that $-\pi/2 < (\chi t + \phi - q_0\pi) \le \pi/2$. As the factor $c(\phi,0)$ defined in equation (56) is peaked around $\phi = 0$, we neglect the dependence of q_0 on ϕ so that the integer q_0 is finally chosen such that $-\pi/2 < (\chi t - q_0\pi) \le \pi/2$. In this way we obtain

$$\langle\psi(t)|c_a^\dagger c_b|\psi(t)\rangle = (-1)^{q_0(N-1)}\frac{\tilde{N}}{2}e^{i(\chi\alpha t - 2D)} \times \int_{-\pi/2}^{\pi/2}\frac{d\phi}{\pi}c(\phi,0)c^*(\phi+\chi t - q_0\pi, 0)\, e^{-i(2\phi+\chi t - q_0\pi)}. \qquad (60)$$

The next step is to average the factor $e^{i(\chi\alpha t-2D)}$over the stochastic realizations. The procedure closely follows the one in the Appendix A. By identifying $\tilde{N}$ with N, as in the previous case, and by extending the boundaries of integration in equation (60) to $\pm\infty$ we can express the result in the compact form[2]:

$$\langle c_a^\dagger c_b\rangle^{fix} = \frac{N}{2}e^{-\lambda t[1-u(t)]}\sum_{q=0}^{+\infty} \times e^{-[(\chi t - q\pi)/2]^2/2\Delta\phi^2}(-1)^{q(N-1)} \qquad (62)$$

[2] To obtain equation (62) we use the condition $\Delta\phi \ll 1$ to set:

$$\langle c_a c_b^\dagger\rangle_{t=0}^{fix} = \frac{N}{2}\left(\int_{-\pi/2}^{\pi/2}\frac{d\phi}{\pi}c^2(\phi,0)\,e^{-2i\phi}\right) \simeq \frac{N}{2}\cdot \qquad (61)$$

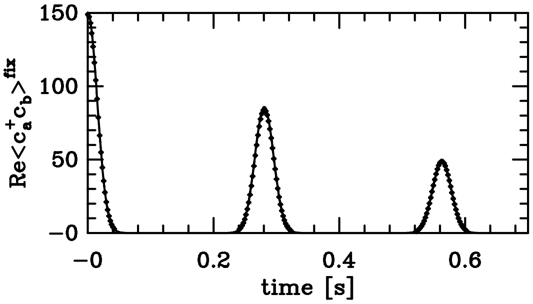

Fig. 3. Collapses and revivals of $\langle c_a^\dagger c_b\rangle^{fix}$ for an initial phase distribution broader than that of the phase state. The initial total number of atoms is $N = 301$. The initial distribution for the difference in the number of particles in the two condensates is Gaussian with a standard deviation $\Delta n = 6$ and a vanishing mean (so that $\bar{N}_a = \bar{N}_b$). The other parameters are as in Figure 2b. Diamonds: numerical result with 2.5×10^4 Monte-Carlo wave functions. Solid line: analytical result.

where $u(t)$ is defined in equation (53). The factor involving the sum over q in equation (62) plays the role of the factor $[\cos(\chi t)]^{N-1}$ in equation (54) which was obtained for an initial phase state. At each time $t_R = q\pi/\chi$ there is a revival of the quantity $\langle c_a^\dagger c_b\rangle^{fix}$ and equation (62) reduces to the very simple expression:

$$\langle c_a^\dagger c_b\rangle^{fix}_{t=t_R} = (-1)^{q(N-1)}\langle c_a^\dagger c_b\rangle^{fix}_{t=0}\, e^{-\lambda t_R}. \qquad (63)$$

This formula does not depend on the initial width $\Delta\phi$ and coincides with the one equation (55) obtained for a phase state. There is therefore no possible way of reducing the damping of the revivals by adjusting the initial width of the phase distribution. Only the temporal width of the revivals is larger for a distribution broader than that for a phase state, as it clearly appears from a comparison between Figure 3 and the previous Figure 2b.

Remark: Formula (62) can also be used to study the collapse of the phase around $t = 0$. For short times ($t \ll t_R$) we expand $u(t)$ to second order in t obtaining:

$$\langle c_a^\dagger c_b\rangle^{fix} \simeq \frac{N}{2}\exp\left\{-\frac{(\chi t)^2}{8\Delta\phi^2}\left[1+\frac{4}{3}m^2\Delta\phi^2\lambda t\right]\right\}. \qquad (64)$$

In the absence of losses we recover the collapse time $t_c = 2\Delta\phi/\chi$ [10] as the half temporal width at the relative height $e^{-1/2}$ of the mean signal $\langle c_a^\dagger c_b\rangle^{fix}$. Losses start *accelerating* the collapse significantly when $\lambda t_c > 1/m^2\Delta\phi^2$. As this last quantity is much larger than 1 the subsequent revivals cannot then be observed.

5 Evolution of the relative phase distribution

We turn now our attention to the phase distribution $|c(\phi)|^2$ which could be reconstructed in an experiment for example *via* a series of multichannel measurements. We show an example of the procedure in Figure 4 [13,20].

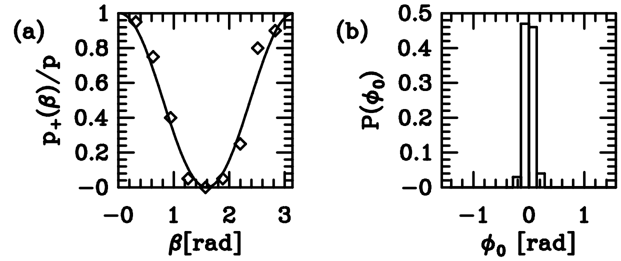

Fig. 4. Monte-Carlo simulation of a multichannel detection experiment using the device in Figure 1 to sample the relative phase distribution corresponding to the initial state of Figure 3. (a) Single realization of the multichannel detection: For each dephasing $\beta_i = i\pi/10, i = 0\ldots 9$ added to one of the input channels of the beam splitter, $p_+(\beta_i)$ (resp. $p_-(\beta_i)$) particles are detected in the + (resp. −) output channel of the beam splitter with $p_+(\beta_i) + p_-(\beta_i) = p = 20$. The obtained integers $p_+(\beta_i)$ (diamonds) are fitted with the function $p\cos^2(\phi_0 - \beta)$ (solid line) where $-\pi/2 < \phi_0 \le \pi/2$ is the adjustable parameter, varying from one realization to the other. (b) After 100 realizations of the multichannel detection (each starting with new condensates): histogram for the obtained values of ϕ_0.

In the frame of our model, the evolution of $c(\phi)$ can be obtained numerically from the evolution of the state vector $|\psi(t)\rangle$ expanded on the Fock state basis by using equation (22); however, as we show in the following, the approximated analytical treatment allows us also in this case to find some simple results at the revival times.

Let the initial state of the condensate, with a total number N of atoms, be characterized by a given relative phase distribution amplitude $c(\phi, 0)$; the state vector at time t is then given by our approximated formula equation (43). One can easily check that the integrand in equation (43) is periodic of period π so that we can shift the interval of integration to obtain[3]:

$$|\psi(t)\rangle = \mathcal{B}(t)e^{-i\chi\hat{n}^2t/4}\int_{-\pi/2}^{\pi/2}\frac{d\phi}{\pi}\,\tilde{c}(\phi - D - v(\tilde{N})t, 0)|\phi\rangle_{\tilde{N}} \qquad (65)$$

where $\tilde{c}(\phi) = e^{-i\alpha\phi}c(\phi)$ and $\tilde{N} = N - mk$. This result has a very suggestive interpretation: the loss processes in a single realization *shift* the relative phase distribution by a random amount D, and the overall evolution can be separated in a random shift plus the Hamiltonian evolution. To make clearer this interpretation, we have plotted in Figure 5 the phase distribution at the second revival time (given by Eq. (35) with $q = 2$) for different realizations. For $\lambda t_R \simeq 1$, as in the figure, there is an important fraction of realizations in which the relative phase is shifted considerably. This is the reason why the relative phase distribution at the revival time will be smeared out by the losses when we take the average over the stochastic realization, which we do now.

[3] When $\phi \to \phi+\pi$, $c(\phi, 0)$ is multiplied by $(-1)^N$, $\exp(-i\alpha\phi)$ is multiplied by $(-1)^{mk}$, and the phase state $|\phi + D + vt\rangle_{\tilde{N}}$ is multiplied by $(-1)^{N-mk}$.

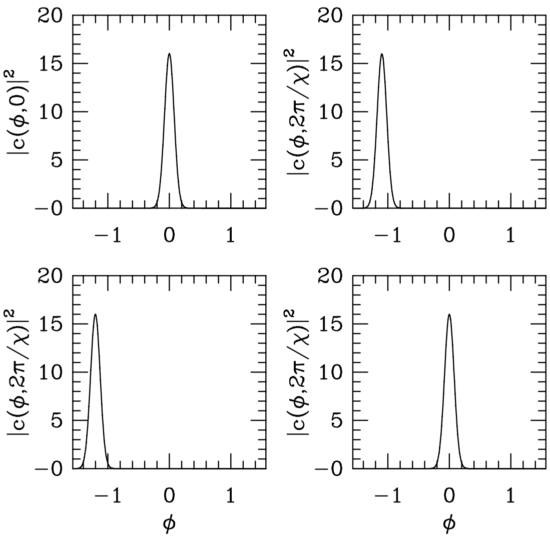

Fig. 5. Single realization relative phase probability distribution at $t = 0$ and at the 2nd revival time $t = 2\pi/\chi$ for three different Monte-Carlo wave functions. The parameters are as in Figure 3. From upper left to lower right the wave functions have experienced 0, 3, 1 and 0 quantum jumps respectively.

As in Section 4 we consider the symmetric case defined by the equations (45, 46, 47). Furthermore we restrict ourselves to the revival times $t = t_R = q\pi/\chi$, q integer (see Eq. (35)). In this case the Hamiltonian evolution operator in equation (65) takes a simple numerical form (see Eqs. (37, 38)) and by comparing equation (65) to equation (21) we can simply read out the phase distribution amplitude $c(\phi, t)$:

$$c(\phi, t_R) = \tilde{c}(\phi_{\tilde{N}} - D, 0), \tag{66}$$

where:

$$\phi_{\tilde{N}} = \phi - q\pi/2 \quad \text{for } \tilde{N} \text{ even} \tag{67}$$

$$\phi_{\tilde{N}} = \phi \quad \text{for } \tilde{N} \text{ odd.} \tag{68}$$

From equation (66) we see again that a single loss event (which can lead to $D \gtrsim \pi$) has a dramatic effect on the phase distribution.

As shown in the Appendix B the phase distribution at the revival times averaged over the stochastic realizations takes the very simple form:

$$\langle |c(\phi, t_R)|^2 \rangle^{fix} = (1 - e^{\lambda t_R}) + e^{-\lambda t_R} |c(\phi_N, 0)|^2. \tag{69}$$

At the revival time the relative phase distribution is "damped" by the factor $e^{-\lambda t_R}$ while a flat background component appears. This effect is clearly shown in Figure 6, where we have plot the averaged relative phase distribution at $t = 0$ and at the second revival time.

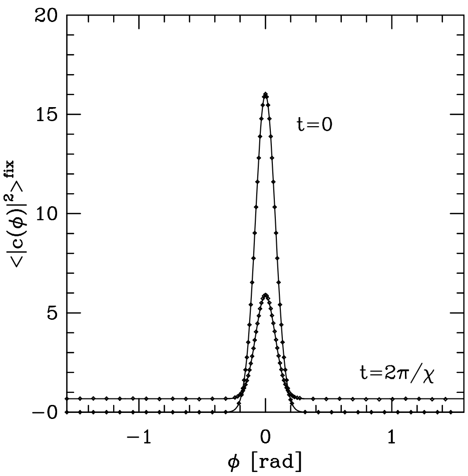

Fig. 6. Relative phase probability distribution at $t = 0$ and at the 2nd revival time. The parameters are as in Figure 3. Solid line: analytical prediction. Diamonds: average of 2.5×10^4 Monte-Carlo wave functions.

6 Effect of an asymmetry between the two condensates

In the previous sections we have investigated the relative phase dynamics in the *symmetric* case for the two condensates. In this section we extend the analysis to account for a small imbalance in the initial average number of particles

$$|\bar{N}_b - \bar{N}_a| \ll \bar{N}, \tag{70}$$

where $\bar{N}$ is the average of the total initial number of particles, and for arbitrary values of the parameters μ_a, μ_b, γ_a, γ_b. We restrict the calculation to the contrast of the interference fringes between the two condensates averaged over many experimental realizations, assuming an initial phase distribution broader than the phase state.

Our initial Monte-Carlo wave function has a fixed total number of particles equal to N, and a Gaussian distribution for number of particles in each condensate. The calculation of $\langle c_a^\dagger c_b \rangle^{fix}$ is now slightly more involved than in the symmetric case, as the phase distribution amplitude $c(\phi, 0)$ acquires a phase factor varying rapidly with ϕ at the scale $1/\sqrt{N}$. All the calculations are therefore put in the Appendix C, and we give here the result only at the revival time $t = t_R$:

$$\langle c_a^\dagger c_b \rangle_{t=t_R}^{fix} = (-1)^{q(N-1)} \frac{N}{2} e^{-2iv(N)t_R} e^{-\lambda t_R [1 - U(t_R)]}, \tag{71}$$

where $v(N)$ is defined by equation (34) and $U(t)$ is a function of time (see Eq. (C.11) in Appendix C). In Figure 7 we show an example of the time evolution of $\langle c_a^\dagger c_b \rangle^{fix}$ in the case of a 10% asymmetry in the initial number of particles $\bar{N}_a$ and $\bar{N}_b$. As far as the *damping* of the revivals is concerned, no significant difference appears with respect

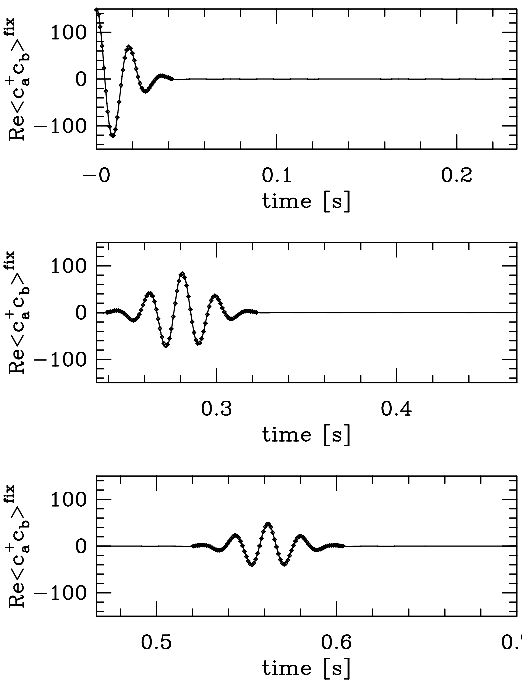

Fig. 7. Collapses and revivals of $\langle c_a^\dagger c_b\rangle^{fix}$ for a 10% asymmetry in the initial number of particles $\bar{N}_a$ and $\bar{N}_b$ in the condensates $\bar{N}_a = 135.5$ and $\bar{N}_b = 165.5$, leading to $\gamma_a \neq \gamma_b$, $\mu_a \neq \mu_b$. The initial total number of atoms is $N = 301$. The initial distribution for the difference in the numbers of particles n in the condensates is Gaussian with a standard deviation $\Delta n = 6$, and a non-vanishing mean value equal to 30. The other parameters are as in Figure 2b. Diamonds: numerical result with 2.5×10^4 Monte-Carlo wave functions. Solid line: analytical result.

to the symmetric case. The damping of the revivals is in this case ruled by the exponent:

$$-\lambda t_R[1 - \mathrm{Re}U(t_R)] \tag{72}$$

where:

$$\mathrm{Re}U(t_R) = \frac{1}{\lambda}\left(\lambda_b\,\mathrm{sinc}(m\mu_b' t_R/\hbar) + \lambda_a\,\mathrm{sinc}(m\mu_a' t_R/\hbar)\right), \tag{73}$$

where $\mathrm{sinc}(x) = \sin(x)/x$. Obviously $|\mathrm{Re}U(t_R)| \leq 1$, meaning that an asymmetry between the condensates cannot *amplify* the revivals with respect to the lossless case. From equation (73) we notice, just as a curiosity, that a complete suppression of the effect of the losses ($\mathrm{Re}U(t_R) = 1$) would occur only in the case in which there are no losses in the condensate **A** ($\lambda_a = 0$) and no elastic interactions in the condensate **B** ($\mu_b' = 0$) (or *vice versa*).

A trivial effect of the asymmetry, evident in Figure 7, is the appearance of *oscillations* of the mean value $\langle c_a^\dagger c_b\rangle^{fix}$ due to the non zero drift velocity of the relative phase of the condensates. We will see in the next section that this effect, harmless at first sight, can have dramatic consequences when we consider the effect of the dispersion in the initial total number of particles N.

7 Effect of fluctuations in the total number of particles

Through all the previous sections in this paper we have chosen an initial state, represented by our initial Monte-Carlo wave function, with a fixed total number of particles in the condensates. The averages that we calculated $\langle ...\rangle^{fix}$ then correspond to the real quantum mechanical averages supposing that the initial total number of atoms is fixed to a value N for any realization of the experiment. In practice it is probably difficult to produce a Fock state for the condensates and the total number of atoms should be governed by some probability distribution $P(N)$. Since we have analytical formulas for the quantities of interest (such as the average $\langle c_a^\dagger c_b\rangle^{fix}$), it is very simple to add a further averaging over N for a given $P(N)$. Suppose for example that the distribution for the initial total number of atoms is a Poissonian distribution of parameter $\bar{N}$. By averaging the result equation (71), valid at the revival times t_R for slightly asymmetric condensates, we get:

$$|\langle c_a^\dagger c_b\rangle_{t=t_R}^{Poiss}| = \frac{\bar{N}}{2} e^{-\lambda t_R[1-\mathrm{Re}U(t_R)]} \times e^{-\bar{N}\left\{\sin^2(\mu_a' t_R/2\hbar)+\sin^2(\mu_b' t_R/2\hbar)\right\}}. \tag{74}$$

The result equation (74) shows that a slight asymmetry between the condensates kills the revivals of $\langle c_a^\dagger c_b\rangle$: the quantity in curly brackets, multiplied by the large number $\bar{N}$, does not vanish in general when $\mu_a' \neq \mu_b'$. This is due to the fact that the drift velocity of the relative phase $v(N)$ in equation (71) depends on the initial total number of particles, giving to $\langle c_a^\dagger c_b\rangle_{t=t_R}^{fix}$ a phase factor of the form:

$$\exp[-2iv(N)t_R] \propto \exp\left[i\,(N - \bar{N})\,\frac{\mu_b' - \mu_a'}{2\hbar}\,t_R\right] = \exp\left[i\,(N - \bar{N})\,\frac{\mu_b' - \mu_a'}{\mu_b' + \mu_a'}\,q\pi\right]. \tag{75}$$

To be able to observe the revivals it is then necessary to be as close as possible to the symmetric conditions in order to satisfy:

$$\frac{\mu_b' - \mu_a'}{\mu_b' + \mu_a'}\Delta N \ll 1, \tag{76}$$

where ΔN is the width of the distribution $P(N)$.

If the *symmetry* between the condensates is perfectly realized, the atom number fluctuations have the simple effect of doubling the revival time. We show an example in Figure 8 where we averaged the result for $\langle c_a^\dagger c_b\rangle^{fix}$ for an initial phase state (Fig. 2) using a Poissonian distribution for $P(N)$. The main effect is the disappearance of the "odd" revivals; this is due to the fact that the amplitude of these odd revivals for N particles is proportional to $[\cos(q\pi)]^{(N-1)} = (-1)^{(N-1)}$ which alternates its sign depending on the parity of N.

In fact it is possible to show that a Poissonian ensemble of phase states is equivalent to a *coherent state* for the two condensates, as long as one calculates the mean values of

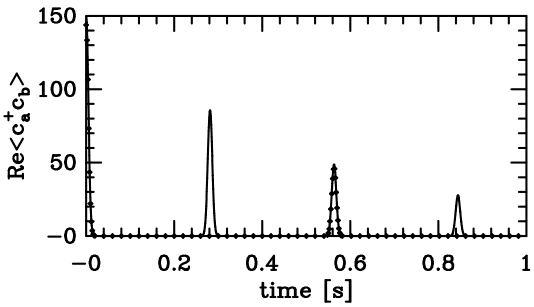

Fig. 8. Collapses and revivals of $\langle c_a^\dagger c_b\rangle^{fix}$ for an initial phase state with $N = 301$ particles (solid line) and after an average over N with a Poisson distribution of parameter $\bar{N} = 301$ (diamonds). The effect of the average is mainly to suppress the odd revivals. The parameters are as in Figure 2b and the results are obtained from the analytical predictions.

operators commuting with the total number of particles in the condensates. For the perfectly symmetric case in Figure 8 we then recover the result obtained in [19] (in the absence of losses) *i.e.* the doubling of the revival period for a coherent state of the condensates as compared to the phase state.

Within the coherent states pictures we can also reinterpret the result equation (74) for the asymmetric case in the following way: in order to observe a revival of the relative phase between two condensates it is necessary that *both* condensates display a phase revival at the same time *i.e.* $\mu_a'/2\hbar t_R = q\pi$ and $\mu_b'/2\hbar t_R = q'\pi$, with q, q' integers.

8 Conclusion

We have studied the dynamics of the relative phase between two Bose-Einstein condensates in presence of m-body loss processes in order to question the observability of the *collapses* and *revivals* of the phase predicted by purely Hamiltonian models.

We have shown that the losses damp exponentially in time the phase dependent quantity $\langle c_a^\dagger c_b\rangle$ (see Eq. (55) for an initial phase state and Eq. (63) for an initially broader phase distribution). The decay rate λ of $\langle c_a^\dagger c_b\rangle$ coincides (up to the factor m) with the mean total number of particles lost per unit of time, and it is therefore approximately N times larger than the inverse lifetime of a particle in the condensates, where N is the total number of particles initially in the condensates.

The dramatic effect of the losses on the relative phase has been suggestively interpreted within the Monte-Carlo wave function approach. In a single realization each single loss event occurring at a time of the order of the revival time shifts the relative phase by a random amount of the order of π. A few loss processes are then sufficient to smear out the relative phase completely at the revival time when the average over the stochastic realizations is taken. For this reason the experimental observation of the revivals is limited to condensates with a small number of atoms where the condition $\lambda t_R < 1$ (where t_R is the revival time

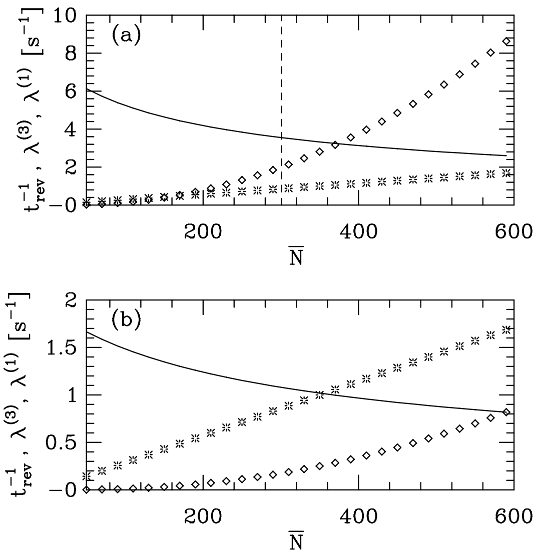

Fig. 9. Collision fluxes $\lambda^{(1)}$ (stars) and $\lambda^{(3)}$ (diamonds), due to one-body and three-body collisions respectively, calculated as in Figure 2, and inverse of the first revival time $1/t_{rev} = \chi/\pi$ (solid line) as a function of the total number of atoms. The trap frequency is $\Omega = 2\pi \times 500$ Hz in (a) and $\Omega = 2\pi \times 200$ Hz in (b). The vertical dashed line for $\bar{N} = 301$ in (a) represents the conditions of Figure 2b. $\lambda^{(1)}$ corresponds to a lifetime due to background gas collisions of 350 seconds.

Eq. (35)) can be satisfied for all the relevant loss processes in the system.

In order to give an idea of the possible scenarios and of the order of magnitudes in different experimental conditions, we have shown in Figure 9 the loss rates due to one-body and three-body collisions and the inverse revival time as functions of the total number of atoms, for two different values of the trap frequencies. For higher trap frequencies (Fig. 9a) the revivals occur on a shorter time scale and one is confronted mainly to three-body losses, while for less confining traps (Fig. 9b) collisions with the residual gas should be taken into account due to longer revival times. Figure 9 shows that phase revivals in presence of losses are in principle observable in condensates with some hundreds of atoms.

By studying the general case of two asymmetric condensates, and the effects of fluctuations in the initial total number of atoms in the condensates, we have finally pointed out a practical difficulty which should be overcome in order to observe phase revivals. The difficulty comes from the fact that in the case of two non perfectly symmetric condensates their relative phase drifts with a velocity *depending* on the initial total number of atoms. For this reason random fluctuations in the initial number of atoms turn out to destroy the relative phase revivals when the asymmetry is too large. A possible way to overcome this problem is of course to use two almost symmetric condensates. Another possibility, which we have not examined in detail, would be to use a condensate **A** which has a collapse time longer than the duration of the experiment ($\bar{N}(\mu'_a t_R/\hbar)^2 \ll 1$) as a *phase reference* to measure the evolving phase of the other condensate **B**.

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Appendix A: Average of the phase factor e^{-2iD}

In this appendix we derive the average over the stochastic realizations of the quantity $e^{-2iD}S(k)$ where D is defined in equation (40) and where $S(k)$ is an arbitrary function of the number of jumps k. We perform the average over the variables δ_{b,ϵ_j} first, using their probability distribution given after equation (28); we have:

$$\begin{aligned}\langle e^{-2iD}\rangle_{\delta_{b,\epsilon_j}} &= \prod_{j=1,k}\frac{1}{\lambda}(\lambda_b e^{-\frac{i}{\hbar}m\mu'_b t_j} + \lambda_a e^{\frac{i}{\hbar}m\mu'_a t_j})\\ &\equiv \prod_{j=1,k} f(t_j). \qquad (A.1)\end{aligned}$$

In order to perform the average over the variables k and τ_j, we need the probability distribution $P_t(k,t_1,t_2,...t_k)$ of having in the time interval $(0,t)$ exactly k jumps separated by time intervals $\tau_j = t_j - t_{j-1}$. Since we assume that the loss processes occur randomly with a constant rate λ, corresponding to a waiting-time distribution of the form $w(\tau) = \lambda e^{-\lambda\tau}$, the probability distribution $P_t(k,t_1,t_2,...t_k)$ is simply [17]:

$$P_t(k,t_1,t_2,....t_k) = \lambda^k e^{-\lambda t}. \qquad (A.2)$$

Using this result we are led to the calculation of a multiple integral of the form:

$$I = \int_{0<t_1<t_2...<t_k<t} f(t_1)f(t_2)...f(t_k)\; dt_1 dt_2...dt_k \qquad (A.3)$$

where $f(t)$ is the argument of the product in equation (A.1). Since I is equal to I_σ calculated for any permutation $t_{\sigma(1)},...t_{\sigma(k)}$ of the integration variables, we can write it as:

$$\begin{aligned}I &= \frac{1}{k!}\left[\sum_\sigma \int_{0<t_{\sigma(1)}<...<t_{\sigma(k)}<t} f(t_1)f(t_2)...f(t_k)dt_1dt_2...dt_k\right]\\ &= \frac{1}{k!}\left[\int_0^t f(t)dt\right]^k. \qquad (A.4)\end{aligned}$$

We then obtain

$$\langle S(k)e^{-2iD}\rangle_{k,\tau_j,\delta_{b,\epsilon_j}} = \sum_{k\geq 0} S(k)\frac{\lambda^k}{k!}\left[\int_0^t f(t)\; dt\right]^k e^{-\lambda t}. \qquad (A.5)$$

In this last equation we may have to introduce by hand a cut-off $N/m-1$ over the index k if $S(k)$ has divergences for $k \geq N/m$ (*i.e.* when no particles are left in the condensates).

Appendix B: Phase distribution at revival times

We are interested in calculating the phase distribution at the revival time averaged over the realizations that is $\langle|c(\phi,t_R)|^2\rangle_{k,\tau_j,\delta_{b,\epsilon_j}}$. We restrict to the symmetric case between the condensates and we start from equation (66). By using equation (22) for $t=0$ we have:

$$\begin{aligned}\langle|c(\phi,t_R)|^2\rangle_{k,\tau_j,\delta_{b,\epsilon_j}} &= |\mathcal{A}(0)|^{-2}\sum_{N_a=0,N}\sum_{N'_a=0,N}\mathrm{fac}(N_a)\\ &\times \mathrm{fac}^*(N'_a)\langle e^{2i(N'_a-N_a)(\phi_{\bar{N}}-D)}\rangle_{k,\tau_j,\delta_{b,\epsilon_j}}\end{aligned} \qquad (B.1)$$

where we have introduced the notation

$$\mathrm{fac}(N_a) = 2^{N/2}\left(\frac{N_a!(N-N_a)!}{N!}\right)^{1/2}\langle N_a, N-N_a|\psi(0)\rangle. \qquad (B.2)$$

The calculation of the average over the stochastic realizations closely resembles the previous one equation (A.1) that we have explained in the Appendix A; we have:

$$\begin{aligned}\langle e^{2i(N'_a-N_a)(\phi_{\bar{N}}-D)}\rangle_{k,\tau_j,\delta_{b,\epsilon_j}} &= \sum_{k\geq 0} e^{-\lambda t_R}\frac{(\lambda t_R)^k}{k!}\\ &\times\left[\frac{\sin[(N'_a-N_a)m\chi t_R]}{(N'_a-N_a)m\chi t_R}\right]^k e^{2i(N'_a-N_a)\phi_{\bar{N}}}. \qquad (B.3)\end{aligned}$$

We note that the terms in the sum in equation (B.3) for $k\neq 0$ are equal to zero unless $(N'_a - N_a) = 0$ in which case the average in equation (B.3) is equal to one. We can then rewrite the result (B.1) as:

$$\begin{aligned}\langle|c(\phi,t_R)|^2\rangle^{fix} &= |\mathcal{A}(0)|^{-2}\Bigg[\sum_{N_a=0,N}\sum_{N'_a=0,N}\delta_{N'_a,N_a}|\mathrm{fac}(N_a)|^2\\ &\times(1-\delta_{N'_a,N_a})\left(\mathrm{fac}(N_a)[\mathrm{fac}(N'_a)]^* e^{2i(N'_a-N_a)\phi_N}e^{-\lambda t_R}\right)\Bigg].\end{aligned} \qquad (B.4)$$

Now by using the property:

$$\sum_{N_a=0,N}|\mathrm{fac}(N_a)|^2|\mathcal{A}(0)|^{-2} = 1 \qquad (B.5)$$

coming from the normalization condition equation (23) and from equation (22), we find the suggestive result equation (69).

Appendix C: Asymmetric condensates

In this appendix we show the explicit calculation of the mean contrast of the interference fringes $\langle c_a^\dagger c_b\rangle^{fix}$ for asymmetric condensates. We consider an initial Monte-Carlo wave function for which the total number of particles N is fixed and the number of particles in condensate **A** has a Gaussian probability distribution:

$$\langle N_a, N-N_a|\psi(0)\rangle = \mathcal{G}e^{-(N_a-x_aN)^2/\Delta n^2} \tag{C.1}$$

where $\mathcal{G}$ is a normalization factor and Δn is the standard deviation for the difference n in the number of particles in the two condensates. The quantities $x_a=\bar{N}_a/(\bar{N}_a+\bar{N}_b)$ and $x_b=\bar{N}_b/(\bar{N}_a+\bar{N}_b)$ are the average fractions of particles initially in the condensate **A** and **B** respectively, which are supposed to be fixed from one realization to the other even in presence of fluctuations of the initial total number of atoms.

We suppose in what follows that

$$1\ll \Delta n \ll \sqrt{N}, \tag{C.2}$$

and

$$|x_aN-x_bN|\ll N. \tag{C.3}$$

We first derive the phase distribution amplitude corresponding to the initial state equation (C.1) by using equation (22). We evaluate the factorials in equation (22) using the Stirling's formula, and we use a local approximation valid for $|N_a-x_aN|\ll\sqrt{N}$:

$$\frac{N_a!(N-N_a)!}{N!}\simeq\frac{(x_aN)!(x_bN)!}{N!}e^{(N_a-x_aN)\ln(x_a/x_b)}. \tag{C.4}$$

By approximating the discrete sum in equation (22) with an integral over N_a ranging from $-\infty$ to $+\infty$ we obtain:

$$c(\phi,0)=\mathcal{N}e^{-\phi^2\Delta n^2}e^{i\kappa\phi} \tag{C.5}$$

where:

$$\kappa=(x_b-x_a)N-\frac{1}{2}\Delta n^2\ln(x_a/x_b) \tag{C.6}$$

and where $\mathcal{N}$ is a normalization factor obtained from equation (23). We note that in the symmetric case $\bar{N}_a=\bar{N}_b$, we recover the Gaussian dependence for $c(\phi)$ of equation (56) with $\Delta n\Delta\phi=1/2$.

We are now ready to calculate $\langle c_a^\dagger c_b\rangle^{fix}$ starting from equation (43). The calculation closely follows the one in Section 4. In particular we use the key property equation (49) to obtain:

$$\begin{aligned}\langle\psi(t)|c_a^\dagger c_b|\psi(t)\rangle &= \frac{1}{\pi^2}|\mathcal{B}(t)|^2|\mathcal{N}|^2\int_{-\pi/2}^{\pi/2}d\phi\int_{-\pi/2}^{\pi/2}d\phi' \\ &\times e^{-(\phi^2+\phi'^2)\Delta n^2}e^{i(\kappa-\alpha)(\phi-\phi')}\frac{\tilde{N}}{2} \\ &\times e^{-i[\phi+\phi'+2(D+v(\tilde{N})t)]}{}_{\tilde{N}-1}\langle\phi'-\chi t|\phi\rangle_{\tilde{N}-1}.\end{aligned} \tag{C.7}$$

The phase factor $e^{i\kappa(\phi-\phi')}$ in the integrand varies rapidly with $\phi-\phi'$ at the scale $1/\sqrt{N}$ when $\bar{N}_b-\bar{N}_a$ is larger than $\sqrt{N}$. For this reason we approximate the scalar product between the phase states $|\phi\rangle_{\tilde{N}}$ and $|\phi'\rangle_{\tilde{N}}$ by a Gaussian $\exp(-\tilde{N}(\phi-\phi')^2/2)$ rather than by the δ distribution of Section 4. This leads to the approximation

$${}_{\tilde{N}-1}\langle\phi'-\chi t|\phi\rangle_{\tilde{N}-1}\simeq(-1)^{q_0(\tilde{N}-1)}e^{-(\tilde{N}-1)(\phi'-\phi-\chi t+q_0\pi)^2/2} \tag{C.8}$$

where the integer q_0 is chosen such that $-\pi/2<(\chi t-q_0\pi)\le\pi/2$. By extending the limits of integration over ϕ,ϕ' to $\pm\infty$ in equation (C.7) we are then left with a double Gaussian integral that can be calculated exactly. The result is quite involved but it can be simplified by using the condition (C.3) and equation (C.2). We take the average over the stochastic realizations and we use again equation (C.2) to simplify the result. We calculate the normalization factor $\mathcal{B}(t)$:

$$\begin{aligned}1\simeq{}&\frac{1}{\pi^2}|\mathcal{N}|^2|\mathcal{B}|^2(t)\left(\frac{2\pi}{4}\Delta n^2\right)^{1/2}\left(\frac{2\pi}{\tilde{N}+\Delta n^2}\right)^{1/2}\\ &\times e^{-\frac{1}{2}(\kappa-\alpha)^2/(\tilde{N}+\Delta n^2)}.\end{aligned} \tag{C.9}$$

We finally obtain for the mean contrast of the interference fringes between **A** and **B** as:

$$\begin{aligned}\langle c_a^\dagger c_b\rangle^{fix}\simeq{}& e^{-\lambda t}e^{-2iv(N)t}\sum_{q=0}^{+\infty}e^{-\frac{1}{2}\Delta n^2[(\chi t-q\pi)]^2}(-1)^{q(N-1)}\\ &\times\sum_{k=0}^{N/m-1}\frac{\tilde{N}}{2}e^{-i\kappa(\chi t-q\pi)\frac{\tilde{N}-1}{\Delta n^2+\tilde{N}-1}}\frac{1}{k!}[\lambda tU(t)]^k\end{aligned} \tag{C.10}$$

where the function $U(t)$ is given by:

$$U(t)=\frac{1}{\lambda}\left(\lambda_b\frac{e^{im\mu_b't/\hbar}-1}{im\mu_b't/\hbar}+\lambda_a\frac{e^{-im\mu_a't/\hbar}-1}{-im\mu_a't/\hbar}\right). \tag{C.11}$$

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